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Classical Collision Mechanics

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Abstract: With the present work we pursue three goals. First, we exemplify some questions of theory of science and the corresponding answers (theoreticity, problem of theoretical terms, empirical claim of a theory, Ramsey eliminability of theoretical terms) by the very simple example of classical collision mechanics. Second, the notion of measurement model can be illustrated by this example in a clear way; in particular, we obtain a complete overview of all measurement models. And third, we get a nice example of the notion of reduction of one theory to another, because collision mechanics can be reduced in a simple way to a specialization of classical particle mechanics.

I. Axioms and models

We axiomatize classical collision mechanics (abbreviated by *CCM* in the following) by introducing the set-theoretic predicate 'is a *CCM*'. Entities matching this predicate are called 'models of *CCM*'; the class of all such models – i. e. the extension of the predicate 'is a *CCM*' – is denoted by $M(CCM)$ or M for short. The statements occurring in the following definitions we call 'axioms of the theory'.

D1 x is a CCM (in symbols: $x \in M(CCM)$ or $x \in M$) iff there exist P, t_1, t_2, v and m such that the following holds:[1]

1) $x = \langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, m\rangle$
2) P is a finite set containing at least two elements
3) $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$
4) $v : P \times \{t_1, t_2\} \to \mathbb{R}^3$
5) $m : P \to \mathbb{R}^+$
6) $\Sigma_{p\in P}\ m(p)v(p, t_1) = \Sigma_{p\in P}\ m(p)v(p, t_2)$.

$P \times \{t_1, t_2\}$ is the *Cartesian product* of P with the set $\{t_1, t_2\}$. The notation '$f : X \to Y$' says that f is a function from X to Y. We will assume in the following that P consists of n elements $p_1, ..., p_n$ $(n \geq 2)$, which are always numbered in this way.

Among all models $\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, m\rangle$ of CCM, only those deserve the name *models of classical collision mechanics* in which P, t_1, t_2, v and m have the following

[1] $\mathbb{R}^+$ is the set of positive real numbers, and $\mathbb{R}^3$ the set of triples of real numbers.

meaning: P is a set of particles, and for all particles $p \in P$, $v(p,t)$ is the velocity of p at a given time t_i $(i = 1,2)$. The velocity was given by a vector from $\mathbb{R}^3$ expressing both magnitude and direction. All velocities and both time points t_1 and t_2 are measured relative to an inertial coordination system with spatial Cartesian coordinates x_1, x_2, x_3 and 'absolute' time coordinate t. $m(p_i)$ denotes the *mass* (in the sense of classical physics) of the particle p_i $(1 \leq i \leq n)$. The function m is called the *mass function*.

The equation D1.6) is the so-called law of conservation of momentum. It is convenient to transform this equation as follows. If we write

$v_i := v(p_i, t_2) - v(p_i, t_1)$ and $m_i := m(p_i)$, $1 \leq i \leq n$,

then D1.6) is equivalent to the following equation

(1) $\Sigma_{1 \leq i \leq n}\ m_i v_i = 0.$

v_i is the difference of final and initial velocities of the particle p_i. The transition to another inertial coordinate system is described by a *Galilean transformation*. In this transformation, the quantities m_i and v_i do not change. Equation (1) thus remains the same for any such coordinate system.

II. A first version of the empirical claim

A first problem with such an empirical, axiomatized, physical theory can be formulated as follows. What has such a theory to do with the world at all? The set-theoretical predicate 'is a CCM' can be understood without reference to the physical world; it is sufficient that one understands the language used for its definition, enriched by mathematics and set-theoretical expressions. It is clear that there are countless systems to which the predicate 'is a CCM' can be applied and which have very little to do with collision mechanics. For example, the following entity is a CCM:

$\langle\{3,4\},\{1,2\},\mathbb{R}^+,\mathbb{R}^3,\{\langle\langle 3,t\rangle,\langle t,0,0\rangle\rangle \mid t \in \{1,2\}\} \cup \{\langle\langle 4,t\rangle,\langle -t,0,0\rangle\rangle \mid t \in \{1,2\}\},\{\langle 3,1\rangle,\langle 4,1\rangle\}\rangle.$

This fact suggests to include another component into the theory, namely a set of concrete physical systems to which one usually applies the classical collision mechanics. By 'one' is meant the group of physicists trained in this field, and 'usually' is meant: as described in books, journal articles, lectures and talks. Also we will describe such systems in this way.

We will represent these systems in set-theoretic formulations. One such system is described by a tuple of the form $\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, m\rangle$ which fulfills D1.1) to D1.5). In the following we will call such tuples which satisfy D1.1) - D1.5), but not necessarily D1.6), *potential models of CCM*:

D2 x is a *potential model of CCM* (in symbols: $x \in M_p(CCM)$ or $x \in M_p$) iff there exist P, t_1, t_2, v and m such that:

1) $x = \langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, m \rangle$
2) P is a finite set containing at least two-elements
3) $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$
4) $v : P \times \{t_1, t_2\} \to \mathbb{R}^3$
5) $m : P \to \mathbb{R}^+$.

Those potential models of CCM that are used as applications of the *physical* theory CCM, are collected to a set I^*. This set we call the *set of intended applications of CCM.* We then conceive the theory CCM itself as a tuple T consisting of the 'formal' part M and the 'pragmatic' part I^*:

$$T = \langle M, I^* \rangle.$$

Here, as said, I^* is a subset of M_p.

The specification of I^* cannot be done by a set-theoretic predicate. How should one be able to capture pragmatic relations by a precise definition, if in such a characterization words like 'usually' occur? The term I^* is open for different linguistic expressions. I^* refers to the same piece of reality as the predicate 'is a CCM' does. The investigation of this piece is intented by the physicists.

Given this state of affairs, one is inclined to describe the function of the predicate 'is a CCM' as follows. This predicate serves to make a claim about a part of the world given in the form of I^*, namely the claim that all elements of I^* are models of CCM. Let us call this assertion the *empirical claim* of CCM.* Assuming that CCM, as a theory, has the form $T = \langle M, I^* \rangle$, with $I^* \subseteq M_p$, the empirical claim* can be defined as follows:

D3 The empirical claim* of the theory $T = \langle M, I^* \rangle$ is the proposition $I^* \subseteq M$.

How can we find out whether the empirical claim of theory T is true? Since the proposition 'For all x: if $x \in I^*$, then $x \in M$' begins with 'for all', one can use a very simple method of verification. One simply checks whether all elements of I^* also lie in M, i.e. whether the law of conservation of momentum is valid for those elements, or not. But as already said, we cannot imagine the elements of I^* giving by precise definitions. Nor will we succeed in finding elements of I^* explicitly described anywhere in the literature. By 'explicit' we mean that all components are given precisely, either in the form of lists as in the purely mathematical example given above, or in the form of linguistic characterizations. The typical way to arrive at a well-specified element of I^* is rather the following. One is introduced to a concrete physical system with the hint that this is an intended application. The fact that this system can actually be understood as a potential model in the sense specified above cannot be seen directly, but must be determined in detail.

In the case of quantitative theories, this determination consists in nothing else than many processes of measurement. Thus, the verification of the empirical claim leads to performances of measurements in certain real systems with the aim of verifying or falsifying the regularities asserted by the theory.

One can try to make these measurements explicit by specifying actions or instructions for action, but in doing so one gets into a swamp of details and pragmatics. One can, on the other hand, try to explicate measurement in the language of the relevant theory. This is much easier, and we turn to this possibility.

In the language of the theory CCM, measurements for the two quantities v and m are to be discussed.

III. Measurement models

It will be said that in a real system a measurement is made if the measured value is uniquely determined by known values of the other quantities occurring in the system and by the special experimental arrangement.

In the case of mass, when restricted to CCM, the description of a measurement – more precisely, the description of a system realized during the measurement – is given by two components. First, the description of the measurement contains the specification of velocity values which can be 'produced' and controlled in the experiment. Second, it contains the specification of the experimental arrangement. Since we confine ourselves to the linguistic framework of CCM, this specification cannot consist of a realistic description of an experiment, but only by a formula. This formula describes conditions for velocities which can be used to determine mass values uniquely. These conditions are expressed by a formula[2] $\mathbf{CV}_{a,b,c}(P, \{t_1, t_2\}, v)$, where the three variables a, b, c are instantiated, respectively, by $P, \{t_1, t_2\}$ and v. The word 'uniquely' in the case of the mass function should always mean: 'uniquely except for an proportionality factor'; uniqueness in a stricter sense cannot be demanded here. If one considers instead of the mass function m the appertaining 'proportionality class' $[m]$, which is defined by

$$[m] := \{m' \mid m' : P \to \mathbb{R}^+ \wedge \exists\alpha\forall p(\alpha \in \mathbb{R}^+ \wedge m'(p) = \alpha m(p))\},$$

we can define the *measurement models for mass of CCM* as follows:

D4 x is a *measurement model for mass of CCM* iff:
1) $x = \langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, m\rangle \in M(CCM)$
2) there is a formula $\mathbf{CV}_{a,b,c}$ which contains (besides mathematical constants) only three free variables a, b, c, so that the following holds:
2.1) $\mathbf{CV}_{a,b,c}(P, \{t_1, t_2\}, v)$
2.2) for all $P', t_1', t_2', v', m', m''$ it holds:
if $\langle P', \{t_1', t_2'\}, \mathbb{R}^+, \mathbb{R}^3, v', m'\rangle \in M$ and $\langle P', \{t_1', t_2'\}, \mathbb{R}^+, \mathbb{R}^3, v', m''\rangle \in M$ and $\mathbf{CV}_{a,b,c}(P', \{t_1', t_2'\}, v')$, then $[m'] = [m'']$
3) $x \in I^*$.

The symbols $P', t_1', t_2', v', m', m''$ here have the status of free variables. We use them only for the sake of readability. $\mathbf{CV}_{a,b,c}(P', \{t_1', t_2'\}, v')$ denotes the formula in which

[2] The symbol $\mathbf{C}$ refers to 'conditions' and $\mathbf{V}$ to 'velocity'.

a, b, and c are replaced by P', $\{t'_1, t'_2\}$ and v', respectively. Condition 1) states that x is a model of CCM. 2.1) states that in the system x all the conditions for velocity values are valid, and 2.2) expresses that m is determined by v uniquely – up to a proportionality factor. 3) is the pragmatic component of the notion of *measurement model for mass of* CCM. 3) guarantees that actually only those systems fall under this notion, which can be applied by physicists to measure mass in CCM. In most cases it is unproblematic to understand the formula $\mathbf{CV}_{a,b,c}(P, \{t_1, t_2\}, v)$ as a description of a part of a measuring method, although $\mathbf{CV}_{a,b,c}(P, \{t_1, t_2\}, v)$ does not refer in any way to the actions performed during the measurement, to the instrument(s) used, or to other components.

D5 x is a *measurement model of CCM for measuring velocities after collision* iff:
1) $x = \langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, m\rangle \in M(CCM)$
2) there is a formula[3] $\mathbf{CM}_{a,b,c}$ which, besides mathematical constants, contains only three free variables a, b, c, so that the following holds:
2.1) $\mathbf{CM}_{a,b,c}(P, \{t_1, t_2\}, m)$
2.2) for all $P', t'_1, t'_2, v', v'', m'$ holds: if
$\langle P', \{t'_1, t'_2\}, \mathbb{R}^+, \mathbb{R}^3, v', m'\rangle \in M$ and $\langle P', \{t'_1, t'_2\}, \mathbb{R}^+, \mathbb{R}^3, v'', m'\rangle \in M$ and $\mathbf{CM}_{a,b,c}(P', \{t'_1, t'_2\}, m')$ and $v' |_{P' \times \{t_1\}} = v'' |_{P' \times \{t_1\}}$,
then $v' |_{P' \times \{t_2\}} = v'' |_{P' \times \{t_2\}}$
3) $x \in I^*$.

D5) is the analogue of D4), except that here $v|_{P \times \{t_1\}}$ plays the role of m in D4). '$v|_B$' denotes the restriction of v to a subset B of the domain of v. D5), however, is uninteresting in the highest degree: There are no velocity measurement models in the sense of this definition at all, since the uniqueness condition D5-2.2) can not be fulfilled. We will briefly discuss this at the end of this paper.

IV. Classification of measurement models for mass

In CCM, unlike many other theories, it is relatively easy to obtain a complete overview of all measurement models for mass. Here, of course, it is assumed that one has an overview of the set I^* of intended applications. The question of a general characterization of the measurement models for a certain quantity of a theory T, which we formulate here for the first time, leads in other physical theories to difficult mathematical problems. Precisely because of the simple mathematical relations used in CCM, CCM provides a good example to illustrate the general problem.

To classify the measurement models for mass, we proceed in two steps. First, we ask what the velocity differences v_i of the particles must be for a given number of particles so that (1) has a solution with $m > 0$ (i.e., $m_1 > 0, ..., m_n > 0$). If (1) is solvable with $m > 0$, we say that (1) is *positively solvable.* The second question is then under which circumstances the masses m_i are uniquely determined by the velocity

[3] Here, $\mathbf{M}$ refers to 'mass'.

differences v_i (except for a proportionality factor). In this question we assume that (1) is already positively solvable. Thus, we ask for the additional conditions which, in the case of positive solvability, guarantee the uniqueness of the values m_i. We say that (1) is *uniquely positive solvable* if (1) is positively solvable and all the values m_i are uniquely determined by the values v_i.

The conditions under which (1) is uniquely positive solvable provide formulas characterizing measurement models for m. Thus, an overview of all possible conditions under which (1) is uniquely positive solvable entails the desired classification of measurement models.

T1 (1) is positively solvable iff there is no $u \in \mathbb{R}^3$ with the following property:
$v_i \otimes u \geq 0$ for all $i \in \{1, ..., n\}$ and $v_i \otimes u > 0$ for at least one[4] $k \in \{1, ..., n\}$.

'$\otimes$' here stands for the scalar product in $\mathbb{R}^3$, i.e., $v_i \otimes u = \Sigma_{1 \leq j \leq 3}\ v_i^j u_j$, where v_i^j and u_j are the components of v_i and u, respectively.

T1) becomes graphically plausible if we consider the hyperplane $u := \{v \in \mathbb{R}^3 \mid v \otimes u = 0\}$ and the positive halfspace $u^+ := \{v \in \mathbb{R}^3 \mid v \otimes u > 0\}$ of the vector u. Roughly speaking we can then say: (1) is positively solvable iff there is no hyperplane $H \subseteq \mathbb{R}^3$ such that all v_i $(1 \leq i \leq n)$ lie on one side of H.

This already answers the first question about the positive solvability of (1) in a graphically satisfying way. We now ask whether (1) is uniquely positive solvable.

T2 Let (1) be positively solvable. Then a) and b) are equivalent:
a) The solution is unique.
b) There is no $J \subseteq \{1, ..., n\}$ such that $J \neq \{1, ..., n\}$ and $\Sigma_{j \in J}\ \lambda_j v_j = 0$ is positively solvable.

This uniqueness criterion, while mathematically beautiful, is difficult to apply. To obtain a more feasible criterion, we reformulate (1) into an equation:

$$(2) \qquad (v_1, ..., v_n) \begin{bmatrix} m_1 \\ \cdot \\ \cdot \\ \cdot \\ m_n \end{bmatrix} = 0$$

$(v_1, ..., v_n)$ is to be understood as a $3 \times n$ matrix with the column vectors $v_1, ..., v_n$. We interpret $(v_1, ..., v_n)$ as a linear mapping $\phi : \mathbb{R}^n \to \mathbb{R}^3$ and write

[4] The proofs of the theorems appearing in this paper are compiled in an appendix.

$$m := \begin{bmatrix} m_1 \\ \cdot \\ \cdot \\ \cdot \\ m_n \end{bmatrix}. \text{ Equation (2) is then: } \phi(m) = 0.$$

D6) a) $[[v_1, ..., v_n]] := \{ \Sigma_{1 \le i \le n} \lambda_i v_i \mid \lambda_i \in \mathbb{R}, 1 \le i \le n\}$.
b) $K^n := \{m \in \mathbb{R}^n \mid m_i > 0, 1 \le i \le n\}$ ('positive cone').
c) $L_\phi := K^n \cap ker\ \phi$ ('solution manifold').

$ker\ \phi$ denotes the kernel of ϕ, i.e. set $\{x \mid \phi(x) = 0\}$. Since K^n is an open set of I $\mathbf{R}^n$, L_ϕ is a open subset of $ker\ \phi$ and, in particular, a submanifold of $\mathbb{R}^n$ (where we also call the empty set a 'manifold').[5] If we denote by $rg\ \phi$ the *rank* of ϕ, i.e., the dimension of $[[v_1, ..., v_n]]$, we obtain the new criterion:

T3 Let (1) be positively solvable. Then the following is true:
The solution is unique, exactly if one of the following three cases exist:
a) $n = 2$ and $rg\ \phi = 1$
b) $n = 3$ and $rg\ \phi = 2$
c) $n = 4$ and $rg\ \phi = 3$.

With the help of the theorems, we now obtain the following possibilities of unique positive solvability, where we first distinguish the cases according to the number of particles. We express the cases of unique positive solvability by $\mathbf{A}_i$ $(i = 1, 2, 3)$ in order to emphasize that the respective characterization after transition into the language of set theory would yield a formula $\mathbf{A}_i$, which could describe a measurement model for mass. We do not write these formulas in an explicit way, because this would yield rather complex, mathematical parts of these formulas.

Case 1), $n = 2$. Because of T1) (1) is positively solvable iff there is a $\lambda < 0$, so that $v_1 = \lambda v_2$. If $v_1 \neq 0$ (and therefore also $v_2 \neq 0$), then by T3) the solution is unique. We thus have a first class of measurement models; $\mathbf{A}_1$: $n = 2$ and $v_1 \neq 0$, and there is a $\lambda < 0$ with $v_1 = \lambda v_2$.

Case 2), $n = 3$. Let $V := [[v_1, v_2, v_3]]$. We further distinguish according to the dimension of V:

(a) dim $V = 1$. Because of T1), (1) is positively solvable iff two vectors from V are non-zero and point in opposite directions. Because of $n = 3$ and $rg\ \phi = 1$, uniqueness is not possible.

(b) dim $V = 2$. A positive solution exists if the conditions of T1) are fulfilled. The solution also is then unique, because of T3).

Thus, we have another class of measurement models; $\mathbf{A}_2$: $n = 3$ and dim $V = 2$, and there is no $u \in \mathbb{R}^3$ with $v_i \otimes u \ge 0$ for all $i \le 3$ and $v_k \otimes u > 0$ for at least one

[5]A precise definition of the topological notions of *manifold* and *submanifold* can be found, for example, in (1).

$k \leq 3$.
c) dim $V = 3$. In this case there is no positive solution because the v_i $(1 \leq i \leq 3)$ are linearly independent.
Case 3), $n = 4$. Let $V := [[v_1, v_2, v_3, v_4]]$. Again we distinguish according to the dimension of V:
a) dim $V = 1$: analogously to 2.a).
b) dim $V = 2$: Analogously to 2.b). However, because of T3) there is no uniqueness.
c) dim $V = 3$: A positive solution exists if the conditions of T1) are fulfilled. It is then also unique – because of T3).

This gives again a class of measurement models; $\mathbf{A}_3$: $n = 4$ and dim $V = 3$, and there is no $u \in \mathbb{R}^3$ with $v_i \otimes u \geq 0$ for all $i \leq 4$ and $v_k \otimes u > 0$ for at least one $k \leq 4$.
Case 4), $n \geq 5$: A positive solution exists if the conditions of T1) are fulfilled. However, by the assumptions of T3) uniqueness is not possible.

Since our case distinctions are exhaustive, we have thus obtained a complete knowledge of the measurement models for mass of CCM.

V. Reduction

Classical mechanics is a very 'meager' theory. One can have good reasons to base the mass measurement on CCM, only if the background of the classical particle mechanics (CPM) is regarded. Only then, if on the basis of the relations of forces it gets clear that the momentum conservation law (D1.6) is valid, the function $m : P \rightarrow \mathbb{R}^+$ from CCM really is the *mass* function. The relation between CCM and CPM can be represented by a simple reduction relation which will be defined in the following.

First, however, we have to state the set-theoretic predicates for CPM and a specialization of CPM in which one can show in a simple way that the monentum conservation law holds.

D7 x is a CPM iff[6] there are P, T, s, m, f such that the following holds:[7]
1) $x = \langle P, T, N, \mathbb{R}^+, \mathbb{R}^3, s, m, f \rangle$
2) P is a finite non empty set
3) T is an interval of $\mathbb{R}$
4) $s : P \times T \rightarrow \mathbb{R}^3$, and there is an open interval T' with $T \subseteq T'$ and a mapping $s' : P \times T' \rightarrow \mathbb{R}^3$ which is continuously differentiable in the second argument, and $s' |_{P \times T} = s$
5) $m : P \rightarrow \mathbb{R}^+$
6) $f : P \times T \times N \rightarrow \mathbb{R}^3$
7) for all $p \in P$ and all $t \in T$, $\Sigma_{i \in N}\ f(p, t, i) = m(p)\mathrm{D}^2 s(p, t)$.

Here $\mathrm{D}^2 s(p, t)$ expresses the second derivative of s – including the boundary points t

[6] An intuitive explanation of the predicate 'is a CPM', which we omit here, can be found, for example, in (3).
[7] N is the set of non-negative integers.

of T which are found in T' – in the second argument.

D8 x is an *ACPM* (*classical particle mechanics, in which 'actio equals reaction' holds*) iff there exist P, T, s, m, f such that the following holds:
1) $x = \langle P, T, N, \mathbb{R}^+, \mathbb{R}^3, s, m, f\rangle$
2) x is a CPM
3) there exists a bijective, inverse mapping $\phi : P \times N \to P \times N$ with the following properties:
(a) for all $p, q \in P$, and all $i, j \in N$ it holds: $\phi(p, i) = (q, j) \to p \neq q$
(b) for all $p, q \in P$, $i, j \in N$ and $t \in T$ it holds: $\phi(p, i) = (q, i) \to f(p, t, i) = -f(q, t, i)$.

D8.3) is to be understood as follows: The i-th force acting on a particle p has as its source a particle q, on which the particle q acts by a j-th force, whose source is now the particle p. This relation between the particles and the forces is described by the function ϕ. Condition a) states that no particle exerts a force on itself; and condition b) expresses just what is called the 'actio equals reaction' principle. Instead of 'x is an $ACPM$' we also say 'x is a model of $ACPM$', or simply $x \in M(ACPM)$.

Analogously to the case of CCM, one can interpret potential models and intended applications:

D9 x is a *potential model of* $ACPM$ (in symbols: $x \in M_p(ACPM)$) iff there are P, T, s, m, f such that:
1) $x = \langle P, T, N, \mathbb{R}^+, \mathbb{R}^3, s, m, f\rangle$
2) P is a finite non empty set
3) T is an interval of $\mathbb{R}$
4) $s : P \times T \to \mathbb{R}^3$, and there is an open interval T' with $T \subseteq T'$ and a mapping $s' : P \times T' \to \mathbb{R}^3$ which is continuously differentiable in the second argument and $s' |_{P\times T} = s$
5) $m : P \to \mathbb{R}^+$
6) $f : P \times T \times N \to \mathbb{R}^3$.

An *intended application of* $ACPM$ is a potential model of $ACPM$ which the physicists actually have in mind in an application of $ACPM$.

This relation between the 'basic theory' $ACPM$ and the theory CCM which should be reduced to $ACPM$, is most conveniently expressed by a relation $\rho \subset M_p(CCM) \times M_p(ACPM)$. The following definition results almost inevitably on the basis of the physical interpretation of the theories:

D10 $\langle x, x'\rangle \in \rho$ iff there exist $P, t_1, t_2, v, m, P', T, s, m', f$ so that the following holds:
1) $x = \langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, m\rangle \in M_p(CCM)$ and
$x' = \langle P', T, N, \mathbb{R}^+, \mathbb{R}^3, s, m', f\rangle \in M_p(ACPM)$
2) $P = P'$
3) $\{t_1, t_2\} = T$
4) $v = \mathrm{D}s_{|P\times\{t_1,t_2\}}$
5) $m = m'$.

This definition testifies that the name 'reduction' for the relation between CCM and $ACPM$ is perhaps a little bit exaggerated (if we compare it to the *paradigm* of reduction: the relation between thermodynamics and statistical mechanics). We would be rather inclined to say that CCM is 'contained' in $ACPM$ (whereby 'containing' would still have to be specified). In any case, however, ρ possesses all those properties which are usually expected from a reduction relation (see (3), pp. 144ff.). In the following theorem the three most important properties are given.

T4 a) $\forall x(x \in M_p(CCM) \rightarrow \exists x'(x' \in M_p(ACPM) \land \langle x, x' \rangle \in \rho))$.
b) $\forall x_1 \forall x_2 \forall x'(\langle x_1, x' \rangle \in \rho \land \langle x_2, x' \rangle \in \rho \rightarrow x_1 = x_2)$.
c) $\forall x \forall x'(\langle x, x' \rangle \in \rho \land x' \in M(ACPM) \rightarrow x \in M(CCM))$.

a) expresses – in model-theoretic formulation – that all basic concepts of CCM are 'transferred' into basic concepts of $ACPM$. b) expresses that CCM is not 'more fundamental' than $ACPM$: for every $x' \in M_p(ACPM)$ for which there is an x_1 with $\langle x_1, x' \rangle \in \rho$, there is only *one* such x_1. In contrast, for every $x \in M_p(CCM)$, there may be several x' with $\langle x, x' \rangle \in \rho$. Finally, statement c) has the most substance, it states that the fundamental law of CCM is derivable from the fundamental laws of $ACPM$ and from the reduction relation.

With the help of the reduction ρ one can now at least partially answer the question when a mass measurement in the framework of CCM, which is represented by a measurement model for mass $x \in I^*$, actually represents a measurement of the mass of classical particle mechanics; namely, in any case, when there is an intended application x' of $ACPM$, so that $x' \in M(ACPM)$ and $\langle x, x' \rangle \in \rho$ are true.

VI. Theoreticity

The question of testing the empirical claim of CCM led us to measurements and these to the discussion of measurement models. The notion of measurement model allows us to sharpen a distinction which was made first by Sneed in (2). Sneed points out that it appears that in a theory T there may be a quantity – let us call it q – which leads to the following problem: Any measurement of q is only possible if T is already true. If there exists such a quantity, it is justified to say that this quantity is *theoretical* (*for* T), because all measurements presuppose the theory T. In other words q cannot be used 'directly' to obtain 'observational propositions' which might confirm or refute T.

To make this idea more precise, we make two assumptions. First, we make the assumption that every measurement of a quantity q can be described as a measurement model of *some* theory. Here, the term 'measurement model for measuring the quantity q in the theory T' is defined in an analogous way as we have just demonstrated for the case of CCM and mass. Second, we make the assumption that there is a certain *hierarchy of theories*.[8] Such a hierarchical order can be understood as follows: a theory

[8] This premise is potentially problematic. First, it is extremely vague, and second, one can raise

T_1 *lies below of* theory T_2 in the hierarchy of theories if theory T_2 uses all terms T_1, while T_1 gets along with fewer terms than T_2 (see (3), p. 60).

Using these two premises, the following criterion for T-theoreticity makes sense:

D11 A quantity q of a theory T is called *T-theoretical* if and only if the following holds:
1) there is a measurement model for q in T
2) there are no measurement models for q in theories T' that lie below of T in the hierarchy of theories.

D11) is certainly not quite an adequate specification of Sneed's idea, because there can be theories T_1 and T_2 such that q is T_1-theoretical *and* T_2-theoretical in the sense of D11). It could be that the correctness of T_1 is independent of the correctness of T_2 and vice versa. In this case, q would be *neither* T_1- *nor* T_2-theoretical in the sense of Sneed's criterion. However, we shall disregard this difficulty in what follows and rely solely on the specification D11). The solution of the problem to give a convincing, watertight definition of the term 'T-theoretical' seems to be still far away. Therefore, we have to be satisfied with provisional explicationes.

Applying the criterion D11) to theory CCM, we obtain the following result: m is CCM-theoretical, but v is not. This is quite clear, because there is certainly no theory in the hierarchy of theories before CCM, which says something about mass measurement, while there are of course kinematic 'pre-theories' for CCM, in which methods for velocity measurement – i.e. measurement models for velocity – can be given.

VII. The problem of theoretical terms and the *Ramsey* version of the empirical claim

The existence of theoretical quantities (or terms) leads us to the so-called 'problem of theoretical terms'. This problem arises at least whenever a theory T has a T-theoretical term q, and whenever, apart from T, one has at one's disposal only those theories T' which lie below of T in the hierarchy of theories, i.e. intuitively speaking, whenever T lies 'at the forefront' of scientific development. We want to consider this problem here only by the example of CCM. Thus we assume in the following that, apart from CCM, we can only resort to theories which lie below of CCM in the hierarchy!

The problem arises when we ask how to confirm the empirical claim*: $I^* \subseteq M$. A confirmation consists of course in the proof that some elements of I^* – namely those which one has investigated – are also models. One must check for each such $x \in I^*$

reasonable doubts about whether the idea of a hierarchy does justice to the actual relations that exist between theories. There is much to be said for the assumption that a definitive criterion for 'T-theoretical' cannot be given until these relations have become clear. However, we are still far from that.

whether $x \in M$ is true, or in other words, we must check whether $x \in I^* \rightarrow x \in M$ is true. How can one check this? The answer already hinted at in Sec. II is that one must, at first, find out how x looks like in a more precise way. That is, one must determine exactly the individual components of x, i.e., $P, \{t_1, t_2\}, v$, and m. In our case we must determine whether t_1, t_2, v, and m can be measured. Now, however, m is CCM-theoretical and any measurement of m already presupposes CCM. More precisely, every measurement of m yields a measurement model for mass and this is, according to D4), a model of CCM. Each measurement of the mass m in the system x presupposes therefore that x is already a model. To check whether x is a model we want to check first to measure the components of x, and only then find out whether the components also fulfill the law of conservation of momentum. Thus we got into a circle.

The circle runs, again briefly, as follows. To confirm $I^* \subseteq M$ we must confirm for each $x \in I^*$ whether $x \in M$. To check this statement, i.e. $x \in I^* \rightarrow x \in M$, we must determine the components of x by measurement. However, any measurement of the theoretical quantity m presupposes that x is already a model. So, to check $x \in I^* \rightarrow x \in M$, we must already presuppose that $x \in M$. So the check goes nowhere.

This problem occurs not only in CCM, but in any theory containing theoretical quantities. The reason for this problem lies simply in the definition of theoretical quantities. If every measurement of a quantity presupposes the theory, then one cannot use this quantity for a 'direct' confirmation of the theory. One can use such a quantity at best for theoretical calculations, in which connections are set up between 'directly measurable' quantities.

These considerations already show how one can come to a verifiable empirical claim in spite of theoretical terms. One must formulate the claim differently, namely in such a way that the theoretical terms play a different role in the claim. In the empirical claim* the theoretical quantity m has exactly the same position as the non-theoretical quantity v. Both appear as components of the models and of the intended applications. Therefore, both must first be determined by measurement before we confirm the claim. Since this is not possible for m without questioning the whole verification, one must simply 'throw out' m at a suitable place. The suitable place here is the occurrence in I^*, because for the description of the models we need m in any case. We will therefore simply omit the function m in the description of the intended applications and denote the resulting set by I. From the tuples of the form $\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, m \rangle$ the symbol m is omitted. (We will also call the elements of this set 'intented applications'.) However, an empirical claim of the form '$I \subseteq M$' then becomes impossible, since the sets I and M are disjoint for purely formal reasons.

The idea how to formulate nevertheless an empirical claim in this situation goes back to Ramsey. He suggested to bind the theoretical quantity, i.e. in this case the quantity m, by an existential quantifier. Then, on the one hand, the quantity no longer occurs 'unprotected', i.e. in such a way that it could be determined by measurement;

on the other hand, however, it still occurs, namely in the form of a variable, and can thus be used to formulate the law of conservation of momentum. The so-called Ramsey-sentence arising by existential quantification over m has, for a single system $x = \langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v\rangle$ of I, the form

$$\exists X(\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, X\rangle \in M).$$

Here, again for clarity, we do not use the symbol m but use a neutral symbol X for a variable. The above formula expresses that *there exists* a function X which can be added to the intended application $\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v\rangle$ so that the resulting structure $\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, X\rangle$ is a model. To express the entire empirical claim of the theory in terms of a Ramsey-sentence, one need only quantify the above formula over all systems $\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v\rangle \in I$. This yields the following formula:

For every $\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v\rangle \in I$ there exists a X such that $\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, X\rangle \in M$.

Let us make this modification a little more precise. To do this, we must first introduce a set of potential partial systems in which m does not occur. We begin by simply omitting from the potential models the last component, the mass function. We denote the resulting class of systems by M_{pp}, and the elements of M_{pp} we call *partial potential models*. The argumentation of Sec. II, according to which $I^* \subseteq M_p$ should hold, can now be repeated with I instead of I^* and M_{pp} instead of M_p. Since here in the systems only the theoretical quantity m was omitted, which cannot be measured directly anyway, we can adopt the argumentation of Sec. II without repeating it here. The result is that I is a subset of M_{pp} which we cannot determine precisely, but only 'paradigmatically'.

We summarize all this in a definition, also modifying somewhat the notion for 'theory CCM':

D12 a) If M_p is defined according to D2), let
$M_{pp} := \{\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v\rangle \mid \exists X(\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, X\rangle \in M_p)\}$.

b) By the theory CCM we mean the quadruple $\langle M, M_p, M_{pp}, I\rangle$, where I is the pragmatic part of CCM as explicated above.

c) The *empirical claim of* CCM is the proposition
'For every $\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v\rangle \in I$, there exists an X such that $\langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, X\rangle \in M$'.

When checking the empirical claim defined according to c), circles do no occur anymore. To check the statement '$x \in I \rightarrow \exists X(\langle x, X\rangle \in M)$', the theoretical quantity m, which originally bothered us, does not need to be measured any more. It does not occur in x any more. Among the data or components, which are measured in a real system, the theoretical quantity does not occur any more. At this point the circle described earlier is broken.

VIII. Ramsey eliminability

A much discussed question in connection with theoretical terms is, whether these are 'really necessary'. The analysis of concrete calculations, forecasts, measurements, as well as results from logic (e.g. *Craig*'s theorem) suggest that in fact theoretical terms are in principle superfluous. In the following we want to show by the example of CCM what Ramsey-eliminability exactly means, and that in CCM the mass is indeed Ramsey-eliminable.

Theoretical quantities are in principle not necessary if they are Ramsey-eliminable. And Ramsey-eliminability means that the class of partial potential models which can be added to models can be formally described without the help of theoretical terms. If this can be done, one can characterize a subclass M^* of M_{pp} without theoretical terms, which contains exactly all partial potential models which can be completed to models. The empirical claim of D12-c) is then equivalent to the claim $I \subseteq M^*$, and in this claim no theoretical quantities occur.

For a more precise and general understanding we introduce the notion of a *reduct*. A partial potential model x' is the reduct of a potential model x, if x' arises from x by omitting the theoretical term m in x. For a set X of potential models, let $r(X)$ denote the set of all reducts of models, i.e., the set of all partial potential models arising from models by omitting m. We say that a theory T of the form $\langle M, M_p, M_{pp}, I\rangle$ has no empirical content if $r(M) = M_{pp}$, i.e. if every partial potential model is a reduct of a model. In this case the empirical claim of T is logically true, no matter what the intended applications are. One need only remark that when using the notion of reduct, the empirical claim simply has the form: $I \subseteq r(X)$. Finally, Ramsey-eliminability of the theoretical quantity m means that there is a set $M^* \subseteq M_{pp}$ which can be formally characterized without the aid of theoretical terms $M^* = r(M)$. It is important to note that in CCM, M^* is determined without the theoretical quantity m. This means that, with CCM one must define M^* by a set-theoretic predicate in which m neither occurs nor is used. If we were to commit ourselves only to the requirement of the non-occurrence of m in the description of M^*, this would be insufficient, for one could bring back and use m in quantified form through the back door. One could define M^* as $r(M)$, using m in quantified form, of course. The requirement that m is not used at all, can be made more precise by the syntactic requirement that only $P, \{t_1, t_2\}$, and v occur as free variables in the formula that determines M and that quantification is only done via 'objects of the basic sets' in this formula. Objects of the basic sets here are the elements of the sets $P, \{t_1, t_2\}$, $\mathbb{R}^+$,$\mathbb{R}^3$.

D13 a) x' is the *reduct of* x $(x' = r(x))$ iff $x' = \langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v\rangle \in M_{pp}$ and $x = \langle P, \{t_1, t_2\}, \mathbb{R}^+, \mathbb{R}^3, v, m\rangle \in M_p$.

b) For $X \subseteq M_p$, let $r(X) := \{x' \in M_{pp} \mid \exists x(x \in X \wedge x' = r(x))\}$.

c) CCM has no empirical content iff $r(M) = M_{pp}$.

d) m is Ramsey-eliminable in CCM iff there is a formula with exactly three free variables a, b, c, such that:

1) for all $P, \{t_1, t_2\}, v$: if $\mathbf{CV}_{a,b,c}(P, \{t_1, t_2\}, v)$, then
$\langle P, \{t_1, t_2\}, \mathrm{I\!R}^+, \mathrm{I\!R}^3, v\rangle \in M_{pp}$

2) $\mathbf{CV}_{a,b,c}(P, \{t_1, t_2\}, v)$ contains only quantifiers over elements of
$P, \{t_1, t_2\}, \mathrm{I\!R}^+, \mathrm{I\!R}^3$

3) $\{\langle P, \{t_1, t_2\}, \mathrm{I\!R}^+, \mathrm{I\!R}^3, v\rangle \mid \mathbf{CV}_{a,b,c}(P, \{t_1, t_2\}, v)\} = r(M)$.

The set defined by $\mathbf{CV}_{a,b,c}(P, \{t_1, t_2\}, v)$ in d.3) is (and was above) called M^*.

T5 a) If CCM has no empirical content, then the empirical claim of CCM is logically true.

T6 a) CCM has empirical content.
b) Mass is Ramsey-eliminable in CCM.

IX. Constraints

CCM in the form presented so far is a rather simple theory. Even as a theory of mass measurement, CCM can be used only in a limited way. In general, one would like to determine also masses of particles which are not by chance found in systems described by CCM. To measure the mass of a given particle in the framework of CCM, an experiment with this and other particles must be carried out. One has to construct measurement models for mass of CCM. The claim that one would have measured the mass of this particle with a measurement model for mass, can be maintained, however, only if we explicitly postulate that a mass value is independent of using a special method to measure mass. This would mean that the mass of the particle under discussion would also be the same in other situations. Only then we would have a theory in which the mass of *any* particle could be described and measured. A 'system-independence' would say that a particle always has the same mass, no matter in what system it will be found or measured. We want to extend CCM by corresponding postulates, whereby a substantial tightening of the empirical content occurs. The formal component, which must be added for technical reasons, is called a 'constraint', because it describes kinds of cross-connections between different models or potential models.

We say that a set of potential models X satisfies the constaint for mass, if particles occurring together in two potential models of X will have the same mass. This means that the mass functions occurring in both systems take the same values for both particles. The constraint of CCM is then given by the set of all such sets X that satisfy the mass condition just described.

D14 a) X *satisfies the constraint for* m iff $X \subseteq M_p$ and for all $x, y \in X$ and for all p: if $p \in P^x \cap P^y$, then $m^x(p) = m^y(p)$.
(b) The *constraint* Q *for* CCM is defined by
$Q := \{X | X \text{ satisfies the constraint for } m\}$.

The empirical claim can then be extended to include the constraint. It reads:

$\exists X(I = r(X) \wedge X \in Q \wedge \mathrm{X} \subseteq M)$.

X. Predictions

An important function of empirical theories is to make unambiguous predictions. In this respect, CCM is a completely useless theory. The only conceivable prognosis would concern the velocities after a collision. However, these are, as already mentioned in Sec. III, not unambiguously determinable in CCM. And this means: unambiguously prediction is impossible:

T7 There is no measurement model of CCM for measuring velocities after the collision.

We have already pointed out earlier that CCM is an extremely meager theory compared to CPM. This intuitive judgment can now be specified in such a way that at least the following points could be mentioned, which show the greater strength and richness of the CPM. First, CCM is reducible to the CPM – more precisely: to the specialization $ACPM$ – while a reduction in the other direction is not possible. Second, CCM alone does not allow for any unambiguous predictions at all. CPM, on the other hand, with its many specializations, leads to many and extremely fruitful predictions. One of the reasons for the latter is that many special laws can be built into the CPM, which then create a whole network of relationships. However, this is an aspect that will not be discussed further in this paper.

Appendix

Proof of T1:

'⇒': Suppose there is a $u \in \mathbb{R}^3$ with the property in question. Then for any $v = \Sigma_{1\leq i\leq n}\ \lambda_i v_i$ with $\lambda_i > 0$ $(1 \leq i \leq n)$: $v \otimes u = \Sigma_{1\leq i\leq n}\ \lambda_i v_i \otimes u > 0$. Then v cannot be zero and (1) consequently cannot be positively solvable. '⇐': see [4] Corollary 1A.

Proof of T2:

'a) ⇒ b)': Without restriction of generality, let $\Sigma_{1\leq j\leq k}\ \mu_j v_j = 0$, $\mu_j > 0$ $(1 \leq j \leq k)$ and $k < n$. Then it holds: $\Sigma_{1\leq j\leq k}\ (m_j + \mu_j)v_j + \Sigma_{k+1\leq j\leq n}\ m_j v_j =$
$\Sigma_{1\leq i\leq n}\ m_i v_i + \Sigma_{1\leq j\leq k}\ \mu_j v_j = 0$. Thus (1) is not uniquely solvable in the positive.

'(b) ⇐ (a)': Suppose (1) is not uniquely positive solvable. Then there exist $m_i > 0$, $m'_i > 0$ $(1 \leq i \leq n)$ with $\Sigma_{1\leq i\leq n}\ m_i v_i = 0$, $\Sigma_{1\leq i\leq n}\ m'_i v_i = 0$, and no $\lambda \in \mathbb{R}$ with $m'_i = \lambda m_i$ $(1 \leq i \leq n)$. It is clear that there exists a k with $m_k/m'_k \geq m_i/m'_i$ for $i = 1, ..., n$. From this follows:

$$\begin{aligned}0 &= (m_k/m'_k)\Sigma_{1\leq i\leq n}\ m'_i v_i - \Sigma_{1\leq i\leq n} m_i v_i\\ &= \Sigma_{1\leq i\leq n}\ ((m_k/m'_k)m'_i - m_i)v_i\\ &= \Sigma_{1\leq i\leq n, i\neq k}((m_k/m'_k)m'_i - m_i)v_i.\end{aligned}$$

We set $\mu_i := (m_k/m'_k)m'_i - m_i$.

It is clear that $\mu_i \geq 0$ $(1 \leq i \leq n, i \neq k)$, and because of the nonexistence of the proportionality factor λ, at least one $\mu_i > 0$. Consequently, there is a $J \subseteq \{1, ...n\}$ with $J \neq \{1, ..., n\}$ such that $\mu_i > 0$ for all $j \in J$ and $\Sigma_{j \in J}\ \mu_i v_j = 0$.

Proof of T3:

(1) is uniquely positive solvable exactly when the dimension of the manifold L_ϕ is equal to 1: dim $L_\phi = 1$. Since L_ϕ is an open subset of $ker\ \phi$, dim $L_\phi =$ dim $ker\ \phi$ holds. I.e.
(1) is uniquely positive solvable exactly when dim $ker\ \phi = 1$. Because of the formula $ker\ \phi + rg\ \phi = n$ (linear algebra) this is exactly the case when
$rg\ \phi = n - 1$. Because of $0 \leq rg\ \phi \leq 3$ and $n \geq 2$ the assertion follows.

Proof of T4:

(a) and (b) are trivial.
(c): Because of condition D8.3), all forces cancel in pairs so that: $\Sigma_{p \in P, i \in N}\ f(p, t, i) = 0$ for all $t \in T$.
Because of D7.7) then also $\Sigma_{p \in P}\ m(p)\mathrm{D}^2 s(p, t) = 0$, and it follows:
$0 = \int_{t_1}^{t_2} \Sigma_{p \in P}\ m(p)\mathrm{D}^2 s(p, t)\ \mathrm{dt} = \Sigma_{p \in P}\ m(p) \int_{t_1}^{t_2} \mathrm{D}v(p, t)\ \mathrm{dt} =$
$\Sigma_{p \in P}\ m(p)(v(p, t_2) - v(p, t_1))$.

Proof of T5 and T6:

T5 and T6a are trivial. T6.b) is clear because of Theorem T1.

Proof of T7:

The question is whether in equation $\Sigma_{p \in P}\ m(p)v(p, t_1) = \Sigma_{p \in P}\ m(p)v(p_2)$, for all $p \in P$, the value $v(p, t_2)$ can be uniquely determined by the remaining values. As one can easily reflect upon, this leads to the question whether in an equation of the form $m_1 v_1 + m_2 v_2 = m_1 x_1 + m_2 x_2$ $(m_i, v_i, x_i \in \mathbb{R},\ i = 1, 2)$ the values of x_1, x_2 can be uniquely determined by the remaining values. However, it is clear this is not the case for all choices of $m_1 > 0, m_2 > 0$ and v_1, v_2.

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